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A PRESENT VALUE FORMULATION OF THE CLASSICAL EOQ PROBLEM*

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ABSTRACT

The usual analysis of the deterministic economic order quantity problem seeks to minimize the average cost of inventory ordering and holding costs per unit time. An alternative approach described in this paper examines the present value of discounted costs over an infinite horizon. Differences in the solutions and implications of errors using the two methodologies are discussed.

Introduction

The intention of this paper is to make a minor correction in the logical basis of the classical economic order quantity determination formula and to investigate the robustness of the total inventory cost function as compared to that of the classical analysis. The unusual treatment of the financial aspects of the problem is to assume a constant "holding cost," h, per unit time. The classical analysis seeks to minimize total cost per unit time. We will treat inventory as a form of investment that serves as part of the financial structure of the firm, considering an infinite planning horizon and using a constant, continuous-time interest rate, ρ. The problem is then to minimize the present value of the future costs of production and interest.

The Classical Analysis[1]

We assume a constant known product demand rate r given in dollars per unit time and linear production costs including a fixed setup cost of s dollars and a variable order cost of v dollars per unit ordered. In addition, instantaneous delivery, unlimited processing capability, and infinite planning horizon are assumed (Figure 1). The length of time between orders is q/r. Then the total cost per unit time over one order interval for an order of q units is

$$\left[s + vq + \frac{hq^2}{2r}\right]\frac{r}{q} = \frac{rs}{q} + \frac{hq}{2} + vr$$

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[1]Any introductory production text will discuss the classical EOQ formulas.

and the value of q which minimizes total cost is

$$q^{\circ} = \sqrt{\frac{2rs}{h}}$$

The cost per unit time is found by substitution to be

$$\frac{rs}{q^{\circ}} + \frac{hq^{\circ}}{2} + vr = \sqrt{2rsh} + vr.$$

Present Value Analysis

In the present value analysis we let $C(q)$ be the present value of the infinite horizon problem following a policy of ordering q for each lot whenever the inventory is used up. Then $C(q) = s + vq + e^{-\rho q/r}C(q)$, which includes the cost of the first order $(s + vq)$ plus the discounted cost of continuing the policy after the length of time q/r when inventory must again be replenished. Therefore, $C(q) = \dfrac{s + vq}{1 - e^{-wq}}$, where we let $w = \rho/r$. The object is to find q to minimize $C(q)$.

The local minimum[2] is found at the point at which the derivative of $C(q)$ vanishes;

$$\frac{dC(q)}{dq} = \frac{v(1 - e^{-wq}) - (s + vq)we^{-wq}}{(1 - e^{-wq})^2} = 0,$$

from which it follows that

$$e^{wq} = \frac{ws}{v} + 1 + wq.$$

The optimal value q^* is uniquely defined by this equation. It is found by the intersection of a straight line and an increasing exponential. The existence of such a positive q^* is assured if w, s, and v are positive. The straight line, $\frac{ws}{v} + 1 + wq$, starts at $(0, 1 + \frac{ws}{v})$ and increases linearly in q. The exponential, e^{wq}, starts at (0,1) and increases, eventually at a much faster rate than linearly in q increases. Hence the straight line initially must lie above the exponential and subsequently after intersection must be below it (Figure 2).

[2]This is a minimum rather than a maximum, since $C(q)$, as the ratio of a linear function to a concave function, is convex.

FIGURE 1

Inventory Behavior with Instantaneous Replenishment

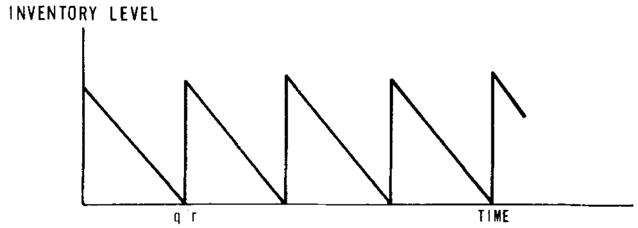


FIGURE 2

Determination of Q*

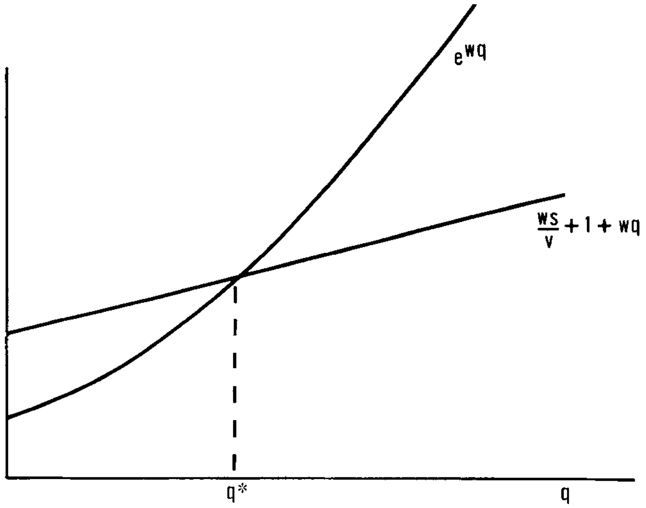
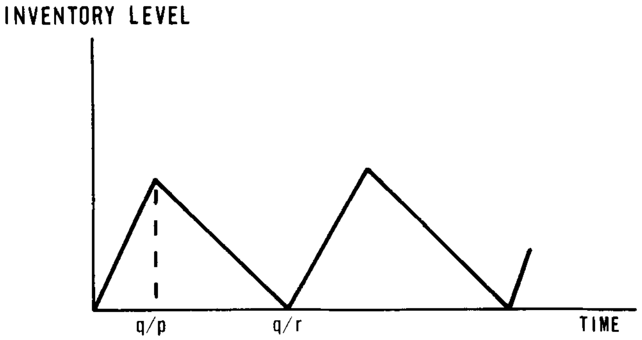


FIGURE 3

Inventory Behavior with Constant Replenishment Rate



We may substitute the Taylor series expansion of the exponential,

$$e^{wq} = 1 + wq + \frac{(wq)^2}{2!} + \frac{(wq)^3}{3!} + \ldots ,$$

into the above equation for determination of q^*, obtaining

$$1 + wq + \frac{(wq)^2}{2!} + \frac{(wq)^3}{3!} \ldots = \frac{ws}{v} + 1 + wq.$$

Letting $\phi(wq)$ represent the terms of higher order than the second in wq, and solving for q^* one obtains

$$q^* = \sqrt{\frac{2s}{wv} - \frac{2\phi(wq)}{w^2}} \leqslant \sqrt{\frac{2s}{wv}}$$

or, equivalently,

$$q^* \leqslant \sqrt{\frac{2rs}{\rho v}} .$$

The right side of this inequality is the same as the classical EOQ formula for q° with $h = \rho v$, the marginal interest cost. In the limit, as $w = \rho/r$ approaches zero, the left and right sides of the above relations approach equality. Although we cannot obtain an expression for q^* which does not involve transcendental functions, we may solve for q^* numerically using the fact that $q^* \leqslant q^{\circ}$, with a method such as Newton's for fast convergence.

It must be emphasized that what is important is not q^*, nor even $C(q^*)$. What is crucial is the error in costs produced by using an erroneous value of q. We must consider in evaluating the model its robustness to errors in the decision variable. To do this we must evaluate the possible proportionate savings

$$S(q) \equiv [C(q) - C(q^*)] \,/\, C(q).$$

This, when worked out, is

$$S(q) = 1 - \frac{(s + vq^*)\,(1 - e^{-wq})}{(s + vq)\,\;(1 - e^{-wq^*})}$$

A Numerical Example

This is an extreme example, chosen to illustrate the magnitude of possible errors in the EOQ and the insensitivity of the costs.

$$s = 5 \cdot 10^3 \qquad h = \rho v = .1$$
$$v = 1 \qquad r = 10$$
$$\rho = .1 \qquad w = \rho/r = .01$$

$$q^{\circ} = \sqrt{\frac{2rs}{h}} = 1{,}000$$

and q^* is such that $e^{wq} = \frac{ws}{v} + 1 + wq$. Hence $q^* = 400$.

We can calculate the cost of any inventory policy using

$$C(q) = \frac{s + vq}{1 - e^{-wq}} .$$

Thus $C(1{,}000) = 6{,}000$ and $C(400) = 5{,}684$. Only 5.3% of the cost is saved by using optimal present value policy $q^* = 400$ instead of $q^{\circ} = 1{,}000$ since $S(q)$ in this case equals .053. Using the classical formula, however, the total cost per unit time for 1,000 units is 100 but for 400 units is 145, which would lead us to believe that employing a 1,000 unit lot size is 31% less costly than employing a 400 unit lot size. The cost function of the present value analysis thus appears to be far less sensitive to errors than that of the classical analysis.

Application of the Present Value Treatment to the Case of Delivery of Batches at a Constant Rate

This model is the same as that in the previous sections except that we assume that delivery of units takes place at a constant rate, p, rather than instantaneously. It is assumed, of course, that $p > r$ (Figure 3).

For the classical analysis the optimal order quantity, q°, becomes

$$\sqrt{\frac{2rs}{h(1 - r/p)}} .$$

The present value analysis provides the following equation to be solved for optimal order quantity q^*:

$$\frac{s}{v}\frac{\rho}{r} + \frac{p}{r} = e^{-\rho q/p}\left(e^{\rho q/r} + \frac{p}{r} - 1\right).$$

As in the case of instantaneous production, such a value of q^* will always exist and be unique.

Conclusion

The logical basis of regarding inventory as an investment and using an infinite horizon for planning seems very sound and proper. However, there are other aspects to inventory holding costs besides those of investment. For example, warehouse space has some opportunity cost in use for other items. Similarly, the costs of production are highly sensitive to shop loads and many transient conditions. However, in the very artificial framework of the classical EOQ problem, the analysis seems preferable to the usual treatment in every elementary textbook on production.

Being logically correct does not, however, imply that the method of calculation should be used. Using the classical method to determine the EOQ seems to have *very little* effect on the total costs (on a percentage basis); hence, other cost considerations not explicitly considered in the model may be more relevant. The classical EOQ cost function is well known for being very *insensitive* to errors in the order quantity. According to the example shown here, it appears that the present value cost function will be even *less* sensitive to errors in the order quantity used.

The conclusion is that very minor cost savings accruing to factors not explicitly considered in the classical model are very likely to predominate over the added costs of using a "non-optimal" EOQ in the framework of this model. The present value model *seems much more robust to errors* in the EOQ than does the classical, average cost per unit time model.